Group Actions (D+F 1.7)

We've already seen several examples of groups whose elements are functions.

e.g. S_n is a set of functions from $\{1,\dots,n\}$ to itself; the elements of D_{2n} are functions from the set of vertices of an n-gon to itself.

The notion of a "group action" generalizes this idea.

Def: A <u>group action</u> of a group G on a set A is a map $G \times A \to A$ (written as $g \cdot a$, for $g \in G$, $a \in A$) such that

1.) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for $g_1, g_2 \in G$, $a \in A$.

2.) $1 \cdot a = a \quad \forall\ a \in A$.

In fact, each element of G is indeed a function from $A \to A$:

for $g \in G$, define $\sigma_g : A \to A$ by $a \mapsto g \cdot a$.

Claim: If G acts on A and $g \in G$, then σ_g is a permutation of A.

Pf: We need to show σ_g is bijective.

Consider $\sigma_{g^{-1}}$. If $a \in A$, then

$$\sigma_{g^{-1}}(\sigma_g(a)) = g^{-1} \cdot (g \cdot a) = 1 \cdot a = a = g \cdot (g^{-1} \cdot a) = \sigma_g(\sigma_{g^{-1}}(a)).$$

i.e. $\sigma_{g^{-1}}$ is an inverse for σ_g. $\square$

Note that S_A is the group consisting of all the permutations of A, so there is a natural relationship between G and S_A:

Claim: If G acts on A, the map $\varphi: G \to S_A$ defined $\varphi(g) = \sigma_g$ is a homomorphism.

Pf: We want to show that for $g_1, g_2 \in G$, $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$. We'll show they're equal on every element of A.

Let $a \in A$.

Then $\varphi(g_1 g_2)(a) = \sigma_{g_1 g_2}(a) = (g_1 g_2)(a) = g_1 \cdot (g_2 \cdot a) = \sigma_{g_1}(\sigma_{g_2}(a))$
$= \varphi(g_1)(\varphi(g_2)(a)) = \varphi(g_1) \circ \varphi(g_2)(a)$. $\square$

Note: When is this homomorphism injective? Exactly when $\ker(G \to S_A) = 1$. i.e. if 1 is the only element that is the identity on A. Equivalently, when no two distinct group elements induce the same permutation on A.

Such a group action is said to be faithful

Ex: 1.) $D_{2n} \times \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a group action. In fact it's faithful: only the identity fixes every vertex.

2.) Consider the vector space $\mathbb{R}^n$. The group $\langle \mathbb{R} - \{0\}, \cdot \rangle$ has a natural group action on $\mathbb{R}^n$ given by the vector space structure: If $a \in \mathbb{R}$, then $a(r_1, \dots, r_n) = (ar_1, ar_2, \dots, ar_n)$.

3.) Define $GL_n(\mathbb{R})$ to be the set of invertible $n \times n$ matrices. This is a group under multiplication (do you see why it's closed?):

- $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the identity
- Everything has an inverse by construction
- matrix multiplication is associative.

$GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by left multiplication:

$$\underset{GL_n(\mathbb{R})}{\begin{bmatrix} & & \\ & & \end{bmatrix}} \cdot \underset{\mathbb{R}^n}{\begin{bmatrix} \\ \end{bmatrix}} = \underset{\text{matrix multiplication}}{\begin{bmatrix} & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix}}$$

4.) Every group acts on itself by multiplication:

$$g \cdot a = ga.$$